

Existence of solutions for a system of fractional boundary value problems

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Abstract. We study the existence of solutions for a system of Riemann-Liouville fractional differential equations with nonlinearities dependent on fractional integrals, supplemented with uncoupled nonlocal boundary conditions which contain various fractional derivatives and Riemann-Stieltjes integrals. We use the fixed point theory in the proof of our main theorems.

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1. Introduction

We consider a nonlinear system of fractional differential equations

$$(S) \quad \begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t), v(t), I_{0+}^{\theta_1} u(t), I_{0+}^{\sigma_1} v(t)) = 0, & t \in (0, 1), \\ D_{0+}^{\beta} v(t) + g(t, u(t), v(t), I_{0+}^{\theta_2} u(t), I_{0+}^{\sigma_2} v(t)) = 0, & t \in (0, 1), \end{cases}$$

with uncoupled nonlocal boundary conditions

$$(BC) \quad \begin{cases} u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & D_{0+}^{\gamma_0} u(1) = \sum_{i=1}^p \int_0^1 D_{0+}^{\gamma_i} u(t) dH_i(t), \\ v(0) = v'(0) = \dots = v^{(m-2)}(0) = 0, & D_{0+}^{\delta_0} v(1) = \sum_{i=1}^q \int_0^1 D_{0+}^{\delta_i} v(t) dK_i(t), \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $n, m \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $\theta_1, \theta_2, \sigma_1, \sigma_2 > 0$, $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, \dots, p$, $0 \leq \gamma_1 < \gamma_2 < \dots < \gamma_p < \alpha - 1$, $\gamma_0 \in [0, \alpha - 1)$, $\delta_i \in \mathbb{R}$ for all $i = 0, \dots, q$, $0 \leq \delta_1 < \delta_2 < \dots < \delta_q < \beta - 1$, $\delta_0 \in [0, \beta - 1)$, D_{0+}^k denotes the Riemann-Liouville derivative of order k (for $k = \alpha, \beta, \gamma_0, \gamma_i, i = 1, \dots, p, \delta_0, \delta_i, i = 1, \dots, q$), I_{0+}^{ζ} is the Riemann-Liouville integral of order ζ (for $\zeta = \theta_1, \sigma_1, \theta_2, \sigma_2$), the functions f and g are nonnegative, and the integrals from the boundary conditions (BC) are Riemann-Stieltjes integrals with H_i for $i = 1, \dots, p$ and K_i for $i = 1, \dots, q$ functions of bounded variation.

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Based on some theorems from the fixed point theory, in this paper we give conditions for the nonlinearities f and g such that problem $(S) - (BC)$ has at least one solution. The fractional equation

$$(E) \quad D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0, 1),$$

with nonlocal boundary conditions

$$(BC_1) \quad u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^p u(1) = \sum_{i=1}^m a_i D_{0+}^q u(\xi_i),$$

where $\xi_i \in \mathbb{R}$, $i = 1, \dots, m$, $0 < \xi_1 < \dots < \xi_m < 1$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$, was investigated in [8]. In [8], the nonlinearity f changes the sign and it is singular in the points $t = 0, 1$, and there the authors used the Guo-Krasnosel'skii fixed point theorem to prove the existence of positive solutions when the parameter λ belongs to various intervals. For some recent results on the existence, nonexistence and multiplicity of positive solutions or solutions for Riemann-Liouville, Caputo or Hadamard fractional differential equations and systems of fractional differential equations subject to various boundary conditions we refer the reader to the monographs [7, 21] and the papers [1, 2, 3, 4, 5, 6, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20].

2. Auxiliary results

We consider the fractional differential equation

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad t \in (0, 1), \quad (1)$$

with boundary conditions

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\gamma_0} u(1) = \sum_{i=1}^p \int_0^1 D_{0+}^{\gamma_i} u(t) dH_i(t), \quad (2)$$

where $h \in C(0, 1) \cap L^1(0, 1)$. We denote by

$$\Delta_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \gamma_0)} - \sum_{i=1}^p \frac{\Gamma(\alpha)}{\Gamma(\alpha - \gamma_i)} \int_0^1 s^{\alpha - \gamma_i - 1} dH_i(s).$$

By standard computations we obtain the following lemma.

Lemma 1. *If $\Delta_1 \neq 0$, then the function $u \in C[0, 1]$ given by*

$$\begin{aligned} u(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{t^{\alpha-1}}{\Delta_1 \Gamma(\alpha - \gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} h(s) ds \\ & - \frac{t^{\alpha-1}}{\Delta_1} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} h(\tau) d\tau \right) dH_i(s), \quad t \in [0, 1], \end{aligned}$$

is a solution of problem (1) – (2).

We also consider the fractional differential equation

$$D_{0+}^{\beta} v(t) + k(t) = 0, \quad t \in (0, 1), \quad (3)$$

with boundary conditions

$$v(0) = v'(0) = \dots = v^{(m-2)}(0) = 0, \quad D_{0+}^{\delta_0} v(1) = \sum_{i=1}^q \int_0^1 D_{0+}^{\delta_i} v(t) dK_i(t), \quad (4)$$

where $k \in C(0, 1) \cap L^1(0, 1)$. We denote by

$$\Delta_2 = \frac{\Gamma(\beta)}{\Gamma(\beta - \delta_0)} - \sum_{i=1}^q \frac{\Gamma(\beta)}{\Gamma(\beta - \delta_i)} \int_0^1 s^{\beta - \delta_i - 1} dK_i(s).$$

Lemma 2. *If $\Delta_2 \neq 0$, then the function $v \in C[0, 1]$ given by*

$$\begin{aligned} v(t) = & -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k(s) ds + \frac{t^{\beta-1}}{\Delta_2 \Gamma(\beta - \delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} k(s) ds \\ & - \frac{t^{\beta-1}}{\Delta_2} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\delta_i-1} k(\tau) d\tau \right) dK_i(s), \quad t \in [0, 1], \end{aligned}$$

is a solution of problem (3) – (4).

Lemma 3 (see [2]). *If $z \in C[0, 1]$ then for $\zeta > 0$ we have*

$$|I_{0+}^{\zeta} z(t)| \leq \frac{\|z\|}{\Gamma(\zeta + 1)}, \quad \forall t \in [0, 1],$$

where $\|z\| = \sup_{t \in [0, 1]} |z(t)|$.

We denote by (I1) the following basic assumptions for problem (S) – (BC) that will be used in the main theorems.

(I1) $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $n, m \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $\theta_1, \theta_2, \sigma_1, \sigma_2 > 0$, $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, \dots, p$, $0 \leq \gamma_1 < \gamma_2 < \dots < \gamma_p < \alpha - 1$, $\gamma_0 \in [0, \alpha - 1)$, $\delta_i \in \mathbb{R}$ for all $i = 0, \dots, q$, $0 \leq \delta_1 < \delta_2 < \dots < \delta_q < \beta - 1$, $\delta_0 \in [0, \beta - 1)$, $H_i : [0, 1] \rightarrow \mathbb{R}$, $i = 1, \dots, p$ and $K_j : [0, 1] \rightarrow \mathbb{R}$, $j = 1, \dots, q$ are functions of bounded variation, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$.

We introduce the following constants:

$$\begin{aligned} M_1 &= 1 + \frac{1}{\Gamma(\theta_1 + 1)}, \quad M_2 = 1 + \frac{1}{\Gamma(\sigma_1 + 1)}, \quad M_3 = 1 + \frac{1}{\Gamma(\theta_2 + 1)}, \\ M_4 &= 1 + \frac{1}{\Gamma(\sigma_2 + 1)}, \quad M_5 = \max\{M_1, M_2\}, \quad M_6 = \max\{M_3, M_4\}, \\ M_7 &= \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{|\Delta_1| \Gamma(\alpha - \gamma_0 + 1)} + \frac{1}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i + 1)} \left| \int_0^1 s^{\alpha - \gamma_i} dH_i(s) \right|, \\ M_9 &= \frac{1}{\Gamma(\beta + 1)} + \frac{1}{|\Delta_2| \Gamma(\beta - \delta_0 + 1)} + \frac{1}{|\Delta_2|} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i + 1)} \left| \int_0^1 s^{\beta - \delta_i} dK_i(s) \right|, \end{aligned}$$

$$M_8 = M_7 - \frac{1}{\Gamma(\alpha + 1)}, \quad M_{10} = M_9 - \frac{1}{\Gamma(\beta + 1)}. \quad (5)$$

We consider the Banach space $X = C[0, 1]$ with supremum norm $\|u\| = \sup_{t \in [0, 1]} |u(t)|$, and the Banach space $Y = X \times X$ with the norm $\|(u, v)\|_Y = \|u\| + \|v\|$. We introduce the operator $A : Y \rightarrow Y$ defined by $A(u, v) = (A_1(u, v), A_2(u, v))$ for $(u, v) \in Y$, where the operators $A_1, A_2 : Y \rightarrow X$ are given by:

$$\begin{aligned} A_1(u, v)(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), v(s), I_{0+}^{\theta_1} u(s), I_{0+}^{\sigma_1} v(s)) ds \\ &\quad + \frac{t^{\alpha-1}}{\Delta_1 \Gamma(\alpha - \gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} f(s, u(s), v(s), I_{0+}^{\theta_1} u(s), I_{0+}^{\sigma_1} v(s)) ds - \frac{t^{\alpha-1}}{\Delta_1} \\ &\quad \times \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} f(\tau, u(\tau), v(\tau), I_{0+}^{\theta_1} u(\tau), I_{0+}^{\sigma_1} v(\tau)) d\tau \right) dH_i(s), \\ A_2(u, v)(t) &= -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, u(s), v(s), I_{0+}^{\theta_2} u(s), I_{0+}^{\sigma_2} v(s)) ds \\ &\quad + \frac{t^{\beta-1}}{\Delta_2 \Gamma(\beta - \delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} g(s, u(s), v(s), I_{0+}^{\theta_2} u(s), I_{0+}^{\sigma_2} v(s)) ds - \frac{t^{\beta-1}}{\Delta_2} \\ &\quad \times \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\delta_i-1} g(\tau, u(\tau), v(\tau), I_{0+}^{\theta_2} u(\tau), I_{0+}^{\sigma_2} v(\tau)) d\tau \right) dK_i(s), \\ &\quad \forall t \in [0, 1], \quad (u, v) \in Y. \end{aligned} \quad (6)$$

By using Lemmas 1 and 2, we note that if (u, v) is a fixed point of operator A , then (u, v) is a solution of problem $(S) - (BC)$.

3. Existence of solutions for $(S) - (BC)$

In this section, we will present some conditions for the nonlinearities f and g such that operator A has at least one fixed point, which is a solution of problem $(S) - (BC)$.

Theorem 1. *Assume that (I1) and*

(I2) *The functions $f, g : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous and there exist $L_1, L_2 > 0$ such that*

$$\begin{aligned} |f(t, x_1, x_2, x_3, x_4) - f(t, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)| &\leq L_1 \sum_{i=1}^4 |x_i - \tilde{x}_i|, \\ |g(t, y_1, y_2, y_3, y_4) - g(t, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)| &\leq L_2 \sum_{i=1}^4 |y_i - \tilde{y}_i|, \end{aligned}$$

for all $t \in [0, 1]$, $x_i, y_i, \tilde{x}_i, \tilde{y}_i \in \mathbb{R}$, $i = 1, \dots, 4$,

hold. If $\Xi := L_1 M_5 M_7 + L_2 M_6 M_9 < 1$, then problem $(S) - (BC)$ has at least one solution $(u(t), v(t))$, $t \in [0, 1]$, where M_5, M_6, M_7, M_9 are given by (5).

Proof. We consider the positive number r given by

$$r = (M_0 M_7 + \widetilde{M}_0 M_9)(1 - L_1 M_5 M_7 - L_2 M_6 M_9)^{-1},$$

where $M_0 = \sup_{t \in [0,1]} |f(t, 0, 0, 0, 0)|$, $\widetilde{M}_0 = \sup_{t \in [0,1]} |g(t, 0, 0, 0, 0)|$. We define the set $\overline{B}_r = \{(u, v) \in Y, \|(u, v)\|_Y \leq r\}$ and first we show that $A(\overline{B}_r) \subset \overline{B}_r$. Let $(u, v) \in \overline{B}_r$. By using (I2) and Lemma 3, for $f(t, u(t), v(t), I_{0+}^{\theta_1} u(t), I_{0+}^{\sigma_1} v(t))$ we deduce the following inequalities:

$$\begin{aligned} & |f(t, u(t), v(t), I_{0+}^{\theta_1} u(t), I_{0+}^{\sigma_1} v(t))| \\ & \leq |f(t, u(t), v(t), I_{0+}^{\theta_1} u(t), I_{0+}^{\sigma_1} v(t)) - f(t, 0, 0, 0, 0)| + |f(t, 0, 0, 0, 0)| \\ & \leq L_1(|u(t)| + |v(t)| + |I_{0+}^{\theta_1} u(t)| + |I_{0+}^{\sigma_1} v(t)|) + M_0 \\ & \leq L_1 \left(\|u\| + \|v\| + \frac{\|u\|}{\Gamma(\theta_1 + 1)} + \frac{\|v\|}{\Gamma(\sigma_1 + 1)} \right) + M_0 \\ & = L_1 \left(\left(1 + \frac{1}{\Gamma(\theta_1 + 1)}\right) \|u\| + \left(1 + \frac{1}{\Gamma(\sigma_1 + 1)}\right) \|v\| \right) + M_0 \\ & = L_1(M_1 \|u\| + M_2 \|v\|) + M_0 \\ & \leq L_1 M_5 \|(u, v)\|_Y + M_0 \leq L_1 M_5 r + M_0, \quad \forall t \in [0, 1]. \end{aligned}$$

In a similar manner, we have

$$\begin{aligned} & |g(t, u(t), v(t), I_{0+}^{\theta_2} u(t), I_{0+}^{\sigma_2} v(t))| \\ & \leq |g(t, u(t), v(t), I_{0+}^{\theta_2} u(t), I_{0+}^{\sigma_2} v(t)) - g(t, 0, 0, 0, 0)| + |g(t, 0, 0, 0, 0)| \\ & \leq L_2(|u(t)| + |v(t)| + |I_{0+}^{\theta_2} u(t)| + |I_{0+}^{\sigma_2} v(t)|) + \widetilde{M}_0 \\ & \leq L_2 \left(\|u\| + \|v\| + \frac{\|u\|}{\Gamma(\theta_2 + 1)} + \frac{\|v\|}{\Gamma(\sigma_2 + 1)} \right) + \widetilde{M}_0 \\ & = L_2 \left(\left(1 + \frac{1}{\Gamma(\theta_2 + 1)}\right) \|u\| + \left(1 + \frac{1}{\Gamma(\sigma_2 + 1)}\right) \|v\| \right) + \widetilde{M}_0 \\ & = L_2(M_3 \|u\| + M_4 \|v\|) + \widetilde{M}_0 \\ & \leq L_2 M_6 \|(u, v)\|_Y + \widetilde{M}_0 \leq L_2 M_6 r + \widetilde{M}_0, \quad \forall t \in [0, 1]. \end{aligned}$$

Then by (6) (the definition of operators A_1 and A_2), we obtain

$$\begin{aligned} |A_1(u, v)(t)| & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (L_1 M_5 r + M_0) ds \\ & \quad + \frac{t^{\alpha-1}}{|\Delta_1| \Gamma(\alpha - \gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} (L_1 M_5 r + M_0) ds \\ & \quad + \frac{t^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} (L_1 M_5 r + M_0) d\tau \right) dH_i(s) \right| \\ & = (L_1 M_5 r + M_0) \left[\frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha-1}}{|\Delta_1| \Gamma(\alpha - \gamma_0 + 1)} \right. \\ & \quad \left. + \frac{t^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i + 1)} \left| \int_0^1 s^{\alpha-\gamma_i} dH_i(s) \right| \right], \quad \forall t \in [0, 1]. \end{aligned}$$

Therefore, we conclude

$$\begin{aligned} \|A_1(u, v)\| &\leq (L_1 M_5 r + M_0) \left[\frac{1}{\Gamma(\alpha + 1)} + \frac{1}{|\Delta_1| \Gamma(\alpha - \gamma_0 + 1)} \right. \\ &\quad \left. + \frac{1}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i + 1)} \left| \int_0^1 s^{\alpha - \gamma_i} dH_i(s) \right| \right] \\ &= (L_1 M_5 r + M_0) M_7. \end{aligned} \quad (7)$$

Arguing as before, we find

$$\begin{aligned} |A_2(u, v)(t)| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (L_2 M_6 r + \widetilde{M}_0) ds \\ &\quad + \frac{t^{\beta-1}}{|\Delta_2| \Gamma(\beta - \delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} (L_2 M_6 r + \widetilde{M}_0) ds \\ &\quad + \frac{t^{\beta-1}}{|\Delta_2|} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\delta_i-1} (L_2 M_6 r + \widetilde{M}_0) d\tau \right) dK_i(s) \right| \\ &= (L_2 M_6 r + \widetilde{M}_0) \left[\frac{t^\beta}{\Gamma(\beta + 1)} + \frac{t^{\beta-1}}{|\Delta_2| \Gamma(\beta - \delta_0 + 1)} \right. \\ &\quad \left. + \frac{t^{\beta-1}}{|\Delta_2|} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i + 1)} \left| \int_0^1 s^{\beta-\delta_i} dK_i(s) \right| \right], \quad \forall t \in [0, 1]. \end{aligned}$$

Then we have

$$\begin{aligned} \|A_2(u, v)\| &\leq (L_2 M_6 r + \widetilde{M}_0) \left[\frac{1}{\Gamma(\beta + 1)} + \frac{1}{|\Delta_2| \Gamma(\beta - \delta_0 + 1)} \right. \\ &\quad \left. + \frac{1}{|\Delta_2|} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i + 1)} \left| \int_0^1 s^{\beta-\delta_i} dK_i(s) \right| \right] \\ &= (L_2 M_6 r + \widetilde{M}_0) M_9. \end{aligned} \quad (8)$$

By relations (7) and (8) we deduce

$$\|A(u, v)\|_Y = \|A_1(u, v)\| + \|A_2(u, v)\| \leq (L_1 M_5 r + M_0) M_7 + (L_2 M_6 r + \widetilde{M}_0) M_9 = r,$$

for all $(u, v) \in \overline{B}_r$, which implies that $A(\overline{B}_r) \subset \overline{B}_r$.

Next, we prove that operator A is a contraction. For $(u_i, v_i) \in \overline{B}_r$, $i = 1, 2$, and for each $t \in [0, 1]$ we obtain

$$\begin{aligned} |A_1(u_1, v_1)(t) - A_1(u_2, v_2)(t)| &\leq \left| -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s, u_1(s), v_1(s), I_{0+}^{\theta_1} u_1(s), I_{0+}^{\sigma_1} v_1(s)) \right. \right. \\ &\quad \left. \left. - f(s, u_2(s), v_2(s), I_{0+}^{\theta_1} u_2(s), I_{0+}^{\sigma_1} v_2(s)) \right] ds \right| \\ &\quad + \frac{t^{\alpha-1}}{|\Delta_1| \Gamma(\alpha - \gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} |f(s, u_1(s), v_1(s), I_{0+}^{\theta_1} u_1(s), I_{0+}^{\sigma_1} v_1(s)) \end{aligned}$$

$$\begin{aligned}
& -f(s, u_2(s), v_2(s), I_{0+}^{\theta_1} u_2(s), I_{0+}^{\sigma_1} v_2(s)) \Big| ds \\
& + \frac{t^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \left| \int_0^1 \left(\int_0^s (s - \tau)^{\alpha - \gamma_i - 1} |f(\tau, u_1(\tau), v_1(\tau), I_{0+}^{\theta_1} u_1(\tau), I_{0+}^{\sigma_1} v_1(\tau)) \right. \right. \\
& \left. \left. - f(\tau, u_2(\tau), v_2(\tau), I_{0+}^{\theta_1} u_2(\tau), I_{0+}^{\sigma_1} v_2(\tau)) \right| d\tau \right) dH_i(s) \Big| \\
& \leq \frac{L_1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \left[|u_1(s) - u_2(s)| + |v_1(s) - v_2(s)| + |I_{0+}^{\theta_1} u_1(s) - I_{0+}^{\theta_1} u_2(s)| \right. \\
& \quad \left. + |I_{0+}^{\sigma_1} v_1(s) - I_{0+}^{\sigma_1} v_2(s)| \right] ds \\
& + \frac{t^{\alpha-1} L_1}{|\Delta_1| \Gamma(\alpha - \gamma_0)} \int_0^1 (1 - s)^{\alpha - \gamma_0 - 1} [|u_1(s) - u_2(s)| + |v_1(s) - v_2(s)| \\
& + |I_{0+}^{\theta_1} u_1(s) - I_{0+}^{\theta_1} u_2(s)| + |I_{0+}^{\sigma_1} v_1(s) - I_{0+}^{\sigma_1} v_2(s)|] ds \\
& + \frac{t^{\alpha-1} L_1}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \left| \int_0^1 \left(\int_0^s (s - \tau)^{\alpha - \gamma_i - 1} [|u_1(\tau) - u_2(\tau)| + |v_1(\tau) - v_2(\tau)| \right. \right. \\
& \left. \left. + |I_{0+}^{\theta_1} u_1(\tau) - I_{0+}^{\theta_1} u_2(\tau)| + |I_{0+}^{\sigma_1} v_1(\tau) - I_{0+}^{\sigma_1} v_2(\tau)| \right] d\tau \right) dH_i(s) \Big| \\
& \leq \frac{L_1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \left[\|u_1 - u_2\| + \|v_1 - v_2\| + \frac{1}{\Gamma(\theta_1 + 1)} \|u_1 - u_2\| \right. \\
& \quad \left. + \frac{1}{\Gamma(\sigma_1 + 1)} \|v_1 - v_2\| \right] ds \\
& + \frac{t^{\alpha-1} L_1}{|\Delta_1| \Gamma(\alpha - \gamma_0)} \int_0^1 (1 - s)^{\alpha - \gamma_0 - 1} \left[\|u_1 - u_2\| + \|v_1 - v_2\| + \frac{1}{\Gamma(\theta_1 + 1)} \|u_1 - u_2\| \right. \\
& \quad \left. + \frac{1}{\Gamma(\sigma_1 + 1)} \|v_1 - v_2\| \right] ds \\
& + \frac{t^{\alpha-1} L_1}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \left| \int_0^1 \left(\int_0^s (s - \tau)^{\alpha - \gamma_i - 1} [\|u_1 - u_2\| + \|v_1 - v_2\| \right. \right. \\
& \quad \left. \left. + \frac{1}{\Gamma(\theta_1 + 1)} \|u_1 - u_2\| + \frac{1}{\Gamma(\sigma_1 + 1)} \|v_1 - v_2\| \right] d\tau \right) dH_i(s) \Big| \\
& \leq \frac{L_1 M_5}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} (\|u_1 - u_2\| + \|v_1 - v_2\|) ds \\
& + \frac{t^{\alpha-1} L_1 M_5}{|\Delta_1| \Gamma(\alpha - \gamma_0)} \int_0^1 (1 - s)^{\alpha - \gamma_0 - 1} (\|u_1 - u_2\| + \|v_1 - v_2\|) ds \\
& + \frac{t^{\alpha-1} L_1 M_5}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \left| \int_0^1 \left(\int_0^s (s - \tau)^{\alpha - \gamma_i - 1} d\tau \right) dH_i(s) \right| \\
& \quad \times (\|u_1 - u_2\| + \|v_1 - v_2\|) \\
& = L_1 M_5 \left(\frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha-1}}{|\Delta_1| \Gamma(\alpha - \gamma_0 + 1)} \right. \\
& \quad \left. + \frac{t^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i + 1)} \left| \int_0^1 s^{\alpha - \gamma_i} dH_i(s) \right| \right) (\|u_1 - u_2\| + \|v_1 - v_2\|).
\end{aligned}$$

Then we conclude

$$\|A_1(u_1, v_1) - A_1(u_2, v_2)\| \leq L_1 M_5 M_7 (\|u_1 - u_2\| + \|v_1 - v_2\|). \quad (9)$$

By similar computation, we also find

$$\|A_2(u_1, v_1) - A_2(u_2, v_2)\| \leq L_2 M_6 M_9 (\|u_1 - u_2\| + \|v_1 - v_2\|). \quad (10)$$

Therefore, by (9) and (10) we obtain

$$\begin{aligned} \|A(u_1, v_1) - A(u_2, v_2)\|_Y &= \|A_1(u_1, v_1) - A_1(u_2, v_2)\| + \|A_2(u_1, v_1) - A_2(u_2, v_2)\| \\ &\leq (L_1 M_5 M_7 + L_2 M_6 M_9) (\|u_1 - u_2\| + \|v_1 - v_2\|) \\ &= \Xi \| (u_1, v_1) - (u_2, v_2) \|_Y. \end{aligned}$$

By using the condition $\Xi < 1$, we deduce that operator A is a contraction. By the Banach contraction mapping principle, we conclude that operator A has a unique fixed point $(u, v) \in \overline{B}_r$, which is a solution of problem $(S) - (BC)$ on $[0, 1]$. \square

Theorem 2. Assume that (I1) and

(I3) The functions $f, g : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous and there exist real constants $c_i, d_i \geq 0, i = 0, \dots, 4$, and at least one of c_0 and d_0 is positive, such that

$$|f(t, x_1, x_2, x_3, x_4)| \leq c_0 + \sum_{i=1}^4 c_i |x_i|, \quad |g(t, y_1, y_2, y_3, y_4)| \leq d_0 + \sum_{i=1}^4 d_i |y_i|,$$

for all $t \in [0, 1], x_i, y_i \in \mathbb{R}, i = 1, \dots, 4$,

hold. If $\Xi_1 := \max\{M_{11}, M_{12}\} < 1$, where $M_{11} = (c_1 + \frac{c_3}{\Gamma(\theta_1+1)})M_7 + (d_1 + \frac{d_3}{\Gamma(\theta_2+1)})M_9$ and $M_{12} = (c_2 + \frac{c_4}{\Gamma(\sigma_1+1)})M_7 + (d_2 + \frac{d_4}{\Gamma(\sigma_2+1)})M_9$, then the boundary value problem $(S) - (BC)$ has at least one solution $(u(t), v(t)), t \in [0, 1]$.

Proof. We prove that operator A is completely continuous. By the continuity of functions f and g we obtain that operators A_1 and A_2 are continuous, and then A is a continuous operator. Next, we prove that A is a compact operator. Let $\Omega \subset Y$ be a bounded set. Then there exist positive constants L_3 and L_4 such that

$$|f(t, u(t), v(t), I_{0+}^{\theta_1} u(t), I_{0+}^{\sigma_1} v(t))| \leq L_3, \quad |g(t, u(t), v(t), I_{0+}^{\theta_2} u(t), I_{0+}^{\sigma_2} v(t))| \leq L_4,$$

for all $(u, v) \in \Omega$ and $t \in [0, 1]$. Therefore, as in the proof of Theorem 1 we deduce that

$$|A_1(u, v)(t)| \leq L_3 M_7, \quad |A_2(u, v)(t)| \leq L_4 M_9, \quad \forall t \in [0, 1], \quad (u, v) \in \Omega.$$

So we obtain

$$\|A_1(u, v)\| \leq L_3 M_7, \quad \|A_2(u, v)\| \leq L_4 M_9, \quad \|A(u, v)\|_Y \leq L_3 M_7 + L_4 M_9, \quad \forall (u, v) \in \Omega,$$

and then $A(\Omega)$ is uniformly bounded.

Next, we will prove that the functions from $A(\Omega)$ are equicontinuous. Let $(u, v) \in \Omega$ and $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
& |A_1(u, v)(t_2) - A_1(u, v)(t_1)| \\
& \leq \left| -\frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s), v(s), I_{0+}^{\theta_1} u(s), I_{0+}^{\sigma_1} v(s)) ds \right. \\
& \quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, u(s), v(s), I_{0+}^{\theta_1} u(s), I_{0+}^{\sigma_1} v(s)) ds \right| \\
& \quad + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta_1| \Gamma(\alpha - \gamma_0)} \left| \int_0^1 (1-s)^{\alpha-\gamma_0-1} f(s, u(s), v(s), I_{0+}^{\theta_1} u(s), I_{0+}^{\sigma_1} v(s)) ds \right| \\
& \quad + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \\
& \quad \times \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} f(\tau, u(\tau), v(\tau), I_{0+}^{\theta_1} u(\tau), I_{0+}^{\sigma_1} v(\tau)) d\tau \right) dH_i(s) \right| \\
& \leq \frac{L_3}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds + \frac{L_3}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
& \quad + \frac{L_3(t_2^{\alpha-1} - t_1^{\alpha-1})}{|\Delta_1| \Gamma(\alpha - \gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} ds + \frac{L_3(t_2^{\alpha-1} - t_1^{\alpha-1})}{|\Delta_1|} \\
& \quad \times \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} d\tau \right) dH_i(s) \right| \\
& = \frac{L_3}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha) + L_3 M_8 (t_2^{\alpha-1} - t_1^{\alpha-1}).
\end{aligned}$$

Then

$$|A_1(u, v)(t_2) - A_1(u, v)(t_1)| \rightarrow 0, \text{ as } t_2 \rightarrow t_1, \text{ uniformly with respect to } (u, v) \in \Omega.$$

In a similar manner, we find

$$|A_2(u, v)(t_2) - A_2(u, v)(t_1)| \leq \frac{L_4}{\Gamma(\beta+1)} (t_2^\beta - t_1^\beta) + L_4 M_{10} (t_2^{\beta-1} - t_1^{\beta-1}),$$

and so

$$|A_2(u, v)(t_2) - A_2(u, v)(t_1)| \rightarrow 0, \text{ as } t_2 \rightarrow t_1, \text{ uniformly with respect to } (u, v) \in \Omega.$$

Thus $A_1(\Omega)$ and $A_2(\Omega)$ are equicontinuous, and then $A(\Omega)$ is also equicontinuous. Hence by the Arzela-Ascoli theorem, we conclude that $A(\Omega)$ is relatively compact, and then A is compact. Therefore, we deduce that A is completely continuous.

We will show next that the set $V = \{(u, v) \in Y, (u, v) = \nu A(u, v), 0 < \nu < 1\}$ is bounded. Let $(u, v) \in V$, that is $(u, v) = \nu A(u, v)$ for some $\nu \in (0, 1)$. Then for any $t \in [0, 1]$ we have $u(t) = \nu A_1(u, v)(t)$, $v(t) = \nu A_2(u, v)(t)$. Hence we find $|u(t)| \leq |A_1(u, v)(t)|$ and $|v(t)| \leq |A_2(u, v)(t)|$ for all $t \in [0, 1]$.

By (I3) we obtain

$$\begin{aligned}
|u(t)| &\leq |A_1(u, v)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[c_0 + c_1|u(s)| + c_2|v(s)| + c_3|I_{0+}^{\theta_1}u(s)| + c_4|I_{0+}^{\sigma_1}v(s)| \right] ds \\
&\quad + \frac{t^{\alpha-1}}{|\Delta_1|\Gamma(\alpha-\gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \left[c_0 + c_1|u(s)| + c_2|v(s)| + c_3|I_{0+}^{\theta_1}u(s)| \right. \\
&\quad \left. + c_4|I_{0+}^{\sigma_1}v(s)| \right] ds \\
&\quad + \frac{t^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha-\gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} [c_0 + c_1|u(\tau)| + c_2|v(\tau)| \right. \right. \\
&\quad \left. \left. + c_3|I_{0+}^{\theta_1}u(\tau)| + c_4|I_{0+}^{\sigma_1}v(\tau)| \right] d\tau \right) dH_i(s) \Big| \\
&\leq \left(c_0 + c_1\|u\| + c_2\|v\| + \frac{c_3}{\Gamma(\theta_1+1)}\|u\| + \frac{c_4}{\Gamma(\sigma_1+1)}\|v\| \right) \left[\frac{t^\alpha}{\Gamma(\alpha+1)} \right. \\
&\quad \left. + \frac{t^{\alpha-1}}{|\Delta_1|\Gamma(\alpha-\gamma_0+1)} + \frac{t^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha-\gamma_i+1)} \left| \int_0^1 s^{\alpha-\gamma_i} dH_i(s) \right| \right].
\end{aligned}$$

Then we deduce

$$\|u\| \leq \left[c_0 + \left(c_1 + \frac{c_3}{\Gamma(\theta_1+1)} \right) \|u\| + \left(c_2 + \frac{c_4}{\Gamma(\sigma_1+1)} \right) \|v\| \right] M_7.$$

In a similar manner, we have

$$\|v\| \leq \left[d_0 + \left(d_1 + \frac{d_3}{\Gamma(\theta_2+1)} \right) \|u\| + \left(d_2 + \frac{d_4}{\Gamma(\sigma_2+1)} \right) \|v\| \right] M_9,$$

and therefore

$$\|(u, v)\|_Y \leq c_0 M_7 + d_0 M_9 + M_{11} \|u\| + M_{12} \|v\| \leq c_0 M_7 + d_0 M_9 + \Xi_1 \|(u, v)\|_Y.$$

Because $\Xi_1 < 1$, we obtain

$$\|(u, v)\|_Y \leq (c_0 M_7 + d_0 M_9)(1 - \Xi_1)^{-1}, \quad \forall (u, v) \in V.$$

So, we conclude that the set V is bounded.

By using the Leray-Schauder alternative theorem, we deduce that operator A has at least one fixed point, which is a solution of our problem $(S) - (BC)$. \square

Theorem 3. Assume that (I1), (I2) and

(I4) There exist the functions $\phi_1, \phi_2 \in C([0, 1], [0, \infty))$ such that

$$|f(t, x_1, x_2, x_3, x_4)| \leq \phi_1(t), \quad |g(t, x_1, x_2, x_3, x_4)| \leq \phi_2(t),$$

for all $t \in [0, 1]$, $x_i \in \mathbb{R}$, $i = 1, \dots, 4$,

hold. If $\Xi_2 := L_1 M_5 \frac{1}{\Gamma(\alpha+1)} + L_2 M_6 \frac{1}{\Gamma(\beta+1)} < 1$, then problem (S) – (BC) has at least one solution on $[0, 1]$.

Proof. We fix $r_1 > 0$ such that $r_1 \geq M_7 \|\phi_1\| + M_9 \|\phi_2\|$. We consider the set $\overline{B}_{r_1} = \{(u, v) \in Y, \|(u, v)\|_Y \leq r_1\}$, and introduce the operators $D = (D_1, D_2) : \overline{B}_{r_1} \rightarrow Y$ and $E = (E_1, E_2) : \overline{B}_{r_1} \rightarrow Y$, where $D_1, D_2, E_1, E_2 : \overline{B}_{r_1} \rightarrow X$ are defined by

$$\begin{aligned} D_1(u, v)(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), v(s), I_{0+}^{\theta_1} u(s), I_{0+}^{\sigma_1} v(s)) ds, \\ E_1(u, v)(t) &= \frac{t^{\alpha-1}}{\Delta_1 \Gamma(\alpha - \gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} f(s, u(s), v(s), I_{0+}^{\theta_1} u(s), I_{0+}^{\sigma_1} v(s)) ds \\ &\quad - \frac{t^{\alpha-1}}{\Delta_1} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \\ &\quad \times \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} f(\tau, u(\tau), v(\tau), I_{0+}^{\theta_1} u(\tau), I_{0+}^{\sigma_1} v(\tau)) d\tau \right) dH_i(s), \\ D_2(u, v)(t) &= -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, u(s), v(s), I_{0+}^{\theta_2} u(s), I_{0+}^{\sigma_2} v(s)) ds, \\ E_2(u, v)(t) &= \frac{t^{\beta-1}}{\Delta_2 \Gamma(\beta - \delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} g(s, u(s), v(s), I_{0+}^{\theta_2} u(s), I_{0+}^{\sigma_2} v(s)) ds \\ &\quad - \frac{t^{\beta-1}}{\Delta_2} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i)} \\ &\quad \times \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\delta_i-1} g(\tau, u(\tau), v(\tau), I_{0+}^{\theta_2} u(\tau), I_{0+}^{\sigma_2} v(\tau)) d\tau \right) dK_i(s), \end{aligned} \quad (11)$$

for $t \in [0, 1]$ and $(u, v) \in \overline{B}_{r_1}$. So $A_1 = D_1 + E_1$, $A_2 = D_2 + E_2$ and $A = D + E$.

By using (I4) for all $(u_1, v_1), (u_2, v_2) \in \overline{B}_{r_1}$ we obtain that

$$\begin{aligned} &\|D(u_1, v_1) + E(u_2, v_2)\|_Y \\ &\leq \|D(u_1, v_1)\|_Y + \|E(u_2, v_2)\|_Y \\ &= \|D_1(u_1, v_1)\| + \|D_2(u_1, v_1)\| + \|E_1(u_2, v_2)\| + \|E_2(u_2, v_2)\| \\ &\leq \frac{1}{\Gamma(\alpha+1)} \|\phi_1\| + \frac{1}{\Gamma(\beta+1)} \|\phi_2\| \\ &\quad + \left(\frac{1}{|\Delta_1| \Gamma(\alpha - \gamma_0 + 1)} + \frac{1}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i + 1)} \left| \int_0^1 s^{\alpha-\gamma_i} dH_i(s) \right| \right) \|\phi_1\| \\ &\quad + \left(\frac{1}{|\Delta_2| \Gamma(\beta - \delta_0 + 1)} + \frac{1}{|\Delta_2|} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i + 1)} \left| \int_0^1 s^{\beta-\delta_i} dK_i(s) \right| \right) \|\phi_2\| \\ &= M_7 \|\phi_1\| + M_9 \|\phi_2\| \leq r_1. \end{aligned}$$

Hence $D(u_1, v_1) + E(u_2, v_2) \in \overline{B}_{r_1}$ for all $(u_1, v_1), (u_2, v_2) \in \overline{B}_{r_1}$.

The operator D is a contraction because

$$\begin{aligned}
& \|D(u_1, v_1) - D(u_2, v_2)\|_Y \\
&= \|D_1(u_1, v_1) - D_1(u_2, v_2)\| + \|D_2(u_1, v_1) - D_2(u_2, v_2)\| \\
&\leq \left(L_1 M_5 \frac{1}{\Gamma(\alpha + 1)} + L_2 M_6 \frac{1}{\Gamma(\beta + 1)} \right) (\|u_1 - u_2\| + \|v_1 - v_2\|) \\
&= \Xi_2 \|(u_1, v_1) - (u_2, v_2)\|_Y,
\end{aligned}$$

for all $(u_1, v_1), (u_2, v_2) \in \overline{B}_{r_1}$, and $\Xi_2 < 1$.

The continuity of f and g implies that operator E is continuous on \overline{B}_{r_1} . We prove in what follows that E is compact. The functions from $E(\overline{B}_{r_1})$ are uniformly bounded because

$$\begin{aligned}
& \|E(u, v)\|_Y \\
&= \|E_1(u, v)\| + \|E_2(u, v)\| \\
&\leq \left(\frac{1}{|\Delta_1| \Gamma(\alpha - \gamma_0 + 1)} + \frac{1}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i + 1)} \left| \int_0^1 s^{\alpha - \gamma_i} dH_i(s) \right| \right) \|\phi_1\| \\
&\quad + \left(\frac{1}{|\Delta_2| \Gamma(\beta - \delta_0 + 1)} + \frac{1}{|\Delta_2|} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i + 1)} \left| \int_0^1 s^{\beta - \delta_i} dK_i(s) \right| \right) \|\phi_2\| \\
&= M_8 \|\phi_1\| + M_{10} \|\phi_2\|, \quad \forall (u, v) \in \overline{B}_{r_1}.
\end{aligned}$$

We prove now that the functions from $E(\overline{B}_{r_1})$ are equicontinuous. We denote by

$$\begin{aligned}
\Psi_{r_1} &= \sup \{ |f(t, u, v, x, y)|, \quad t \in [0, 1], \quad |u| \leq r_1, \quad |v| \leq r_1, \\
&\quad |x| \leq \frac{r_1}{\Gamma(\theta_1 + 1)}, \quad |y| \leq \frac{r_1}{\Gamma(\sigma_1 + 1)} \}, \\
\Theta_{r_1} &= \sup \{ |g(t, u, v, x, y)|, \quad t \in [0, 1], \quad |u| \leq r_1, \quad |v| \leq r_1, \\
&\quad |x| \leq \frac{r_1}{\Gamma(\theta_2 + 1)}, \quad |y| \leq \frac{r_1}{\Gamma(\sigma_2 + 1)} \}.
\end{aligned} \tag{12}$$

Then for $(u, v) \in \overline{B}_{r_1}$, and $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we obtain

$$\begin{aligned}
& |E_1(u, v)(t_2) - E_1(u, v)(t_1)| \\
&\leq \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta_1| \Gamma(\alpha - \gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \Psi_{r_1} ds \\
&\quad + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\gamma_i-1} \Psi_{r_1} d\tau \right) dH_i(s) \right| \\
&\leq \Psi_{r_1} (t_2^{\alpha-1} - t_1^{\alpha-1}) \\
&\quad \times \left[\frac{1}{|\Delta_1| \Gamma(\alpha - \gamma_0 + 1)} + \frac{1}{|\Delta_1|} \sum_{i=1}^p \frac{1}{\Gamma(\alpha - \gamma_i + 1)} \left| \int_0^1 s^{\alpha-\gamma_i} dH_i(s) \right| \right]
\end{aligned}$$

$$\begin{aligned}
&= \Psi_{r_1} M_8 (t_2^{\alpha-1} - t_1^{\alpha-1}), \\
&|E_2(u, v)(t_2) - E_2(u, v)(t_1)| \leq \frac{t_2^{\beta-1} - t_1^{\beta-1}}{|\Delta_2| \Gamma(\beta - \delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} \Theta_{r_1} ds \\
&\quad + \frac{t_2^{\beta-1} - t_1^{\beta-1}}{|\Delta_2|} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\delta_i-1} \Theta_{r_1} d\tau \right) dK_i(s) \right| \\
&\leq \Theta_{r_1} (t_2^{\beta-1} - t_1^{\beta-1}) \\
&\quad \times \left[\frac{1}{|\Delta_2| \Gamma(\beta - \delta_0 + 1)} + \frac{1}{|\Delta_2|} \sum_{i=1}^q \frac{1}{\Gamma(\beta - \delta_i + 1)} \left| \int_0^1 s^{\beta-\delta_i} dK_i(s) \right| \right] \\
&= \Theta_{r_1} M_{10} (t_2^{\beta-1} - t_1^{\beta-1}).
\end{aligned}$$

Hence we find

$$|E_1(u, v)(t_2) - E_1(u, v)(t_1)| \rightarrow 0, \quad |E_2(u, v)(t_2) - E_2(u, v)(t_1)| \rightarrow 0,$$

as $t_2 \rightarrow t_1$ uniformly with respect to $(u, v) \in \overline{B}_{r_1}$. So, $E_1(\overline{B}_{r_1})$ and $E_2(\overline{B}_{r_1})$ are equicontinuous, and then $E(\overline{B}_{r_1})$ is also equicontinuous. By using the Arzela-Ascoli theorem, we deduce that the set $E(\overline{B}_{r_1})$ is relatively compact. Therefore, E is a compact operator on \overline{B}_{r_1} . By the Krasnolsel'skii theorem for the sum of two operators (see [12]), we conclude that there exists a fixed point of operator $D + E (= A)$, which is a solution of problem $(S) - (BC)$. \square

Theorem 4. Assume that (I1), (I2) and (I4) hold. If $\Xi_3 := L_1 M_5 M_8 + L_2 M_6 M_{10} < 1$, then problem $(S) - (BC)$ has at least one solution (u, v) on $[0, 1]$.

Proof. We consider again a positive number $r_1 \geq M_7 \|\phi_1\| + M_9 \|\phi_2\|$, and the operators D and E defined on \overline{B}_{r_1} given by (11). As in the proof of Theorem 3, we obtain that $D(u_1, v_1) + E(u_2, v_2) \in \overline{B}_{r_1}$ for all $(u_1, v_1), (u_2, v_2) \in \overline{B}_{r_1}$.

The operator E is a contraction because

$$\begin{aligned}
&\|E(u_1, v_1) - E(u_2, v_2)\|_Y \\
&= \|E_1(u_1, v_1) - E_1(u_2, v_2)\| + \|E_2(u_1, v_1) - E_2(u_2, v_2)\| \\
&\leq L_1 M_5 M_8 (\|u_1 - u_2\| + \|v_1 - v_2\|) + L_2 M_6 M_{10} (\|u_1 - u_2\| + \|v_1 - v_2\|) \\
&= (L_1 M_5 M_8 + L_2 M_6 M_{10}) \|(u_1, v_1) - (u_2, v_2)\|_Y = \Xi_3 \|(u_1, v_1) - (u_2, v_2)\|_Y,
\end{aligned}$$

for all $(u_1, v_1), (u_2, v_2) \in \overline{B}_{r_1}$, with $\Xi_3 < 1$.

Next, the continuity of f and g implies that operator D is continuous on \overline{B}_{r_1} . We show now that D is compact. The functions from $D(\overline{B}_{r_1})$ are uniformly bounded because

$$\begin{aligned}
\|D(u, v)\|_Y &= \|D_1(u, v)\| + \|D_2(u, v)\| \\
&\leq \frac{1}{\Gamma(\alpha + 1)} \|\phi_1\| + \frac{1}{\Gamma(\beta + 1)} \|\phi_2\|, \quad \forall (u, v) \in \overline{B}_{r_1}.
\end{aligned}$$

Now we prove that $D(\overline{B}_{r_1})$ is equicontinuous. By using Ψ_{r_1} and Θ_{r_1} defined in (12), for $(u, v) \in \overline{B}_{r_1}$ and $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$ we find that

$$\begin{aligned} |D_1(u, v)(t_2) - D_1(u, v)(t_1)| &\leq \frac{\Psi_{r_1}}{\Gamma(\alpha + 1)}(t_2^\alpha - t_1^\alpha), \\ |D_2(u, v)(t_2) - D_2(u, v)(t_1)| &\leq \frac{\Theta_{r_1}}{\Gamma(\beta + 1)}(t_2^\beta - t_1^\beta). \end{aligned}$$

Then we obtain

$$|D_1(u, v)(t_2) - D_1(u, v)(t_1)| \rightarrow 0, \quad |D_2(u, v)(t_2) - D_2(u, v)(t_1)| \rightarrow 0,$$

as $t_2 \rightarrow t_1$ uniformly with respect to $(u, v) \in \overline{B}_{r_1}$. We deduce that $D_1(\overline{B}_{r_1})$ and $D_2(\overline{B}_{r_1})$ are equicontinuous, and so $D(\overline{B}_{r_1})$ is equicontinuous. By using the Arzela-Ascoli theorem, we conclude that the set $D(\overline{B}_{r_1})$ is relatively compact. Then D is a compact operator on \overline{B}_{r_1} . By the Krasnosel'skii theorem we deduce that there exists a fixed point of operator $D + E(= A)$, which is a solution of problem $(S) - (BC)$. \square

Theorem 5. Assume that (I1) and

(I5) The functions $f, g : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous and there exist the constants $a_i \geq 0$, $i = 0, \dots, 4$ with at least one nonzero, the constants b_i , $i = 0, \dots, 4$ with at least one nonzero, and $l_i, m_i \in (0, 1)$, $i = 1, \dots, 4$ such that

$$\begin{aligned} |f(t, x_1, x_2, x_3, x_4)| &\leq a_0 + \sum_{i=1}^4 a_i |x_i|^{l_i}, \\ |g(t, y_1, y_2, y_3, y_4)| &\leq b_0 + \sum_{i=1}^4 b_i |y_i|^{m_i}, \end{aligned}$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$,

hold. Then problem $(S) - (BC)$ has at least one solution.

Proof. Let $\overline{B}_R = \{(u, v) \in Y, \|(u, v)\|_Y \leq R\}$, where

$$\begin{aligned} R \geq \max &\left\{ 10a_0M_7, (10a_1M_7)^{\frac{1}{1-l_1}}, (10a_2M_7)^{\frac{1}{1-l_2}}, \left(\frac{10a_3M_7}{(\Gamma(\theta_1+1))^{l_3}} \right)^{\frac{1}{1-l_3}}, \right. \\ &\left(\frac{10a_4M_7}{(\Gamma(\sigma_1+1))^{l_4}} \right)^{\frac{1}{1-l_4}}, 10b_0M_9, (10b_1M_9)^{\frac{1}{1-m_1}}, (10b_2M_9)^{\frac{1}{1-m_2}}, \\ &\left. \left(\frac{10b_3M_9}{(\Gamma(\theta_2+1))^{m_3}} \right)^{\frac{1}{1-m_3}}, \left(\frac{10b_4M_9}{(\Gamma(\sigma_2+1))^{m_4}} \right)^{\frac{1}{1-m_4}} \right\}. \end{aligned}$$

We prove that $A : \overline{B}_R \rightarrow \overline{B}_R$. For $(u, v) \in \overline{B}_R$, we deduce

$$\begin{aligned} |A_1(u, v)(t)| &\leq \left(a_0 + a_1R^{l_1} + a_2R^{l_2} + a_3 \frac{R^{l_3}}{(\Gamma(\theta_1+1))^{l_3}} + a_4 \frac{R^{l_4}}{(\Gamma(\sigma_1+1))^{l_4}} \right) M_7 \leq \frac{R}{2}, \\ |A_2(u, v)(t)| &\leq \left(b_0 + b_1R^{m_1} + b_2R^{m_2} + b_3 \frac{R^{m_3}}{(\Gamma(\theta_2+1))^{m_3}} + b_4 \frac{R^{m_4}}{(\Gamma(\sigma_2+1))^{m_4}} \right) M_9 \leq \frac{R}{2}, \end{aligned}$$

for all $t \in [0, 1]$. Then we obtain

$$\|A(u, v)\|_Y = \|A_1(u, v)\| + \|A_2(u, v)\| \leq R, \quad \forall (u, v) \in \overline{B}_R,$$

which implies that $A(\overline{B}_R) \subset \overline{B}_R$.

Because the functions f and g are continuous, we conclude that operator A is continuous on \overline{B}_R . In addition, the functions from $A(\overline{B}_R)$ are uniformly bounded and equicontinuous. Indeed, by using the notations (12) with r_1 replaced by R , for any $(u, v) \in \overline{B}_R$ and $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ we find that

$$\begin{aligned} |A_1(u, v)(t_2) - A_1(u, v)(t_1)| &\leq \frac{\Psi_R}{\Gamma(\alpha + 1)}(t_2^\alpha - t_1^\alpha) + \Psi_R M_8(t_2^{\alpha-1} - t_1^{\alpha-1}), \\ |A_2(u, v)(t_2) - A_2(u, v)(t_1)| &\leq \frac{\Theta_R}{\Gamma(\beta + 1)}(t_2^\beta - t_1^\beta) + \Theta_R M_{10}(t_2^{\beta-1} - t_1^{\beta-1}). \end{aligned}$$

Therefore,

$$|A_1(u, v)(t_2) - A_1(u, v)(t_1)| \rightarrow 0, \quad |A_2(u, v)(t_2) - A_2(u, v)(t_1)| \rightarrow 0, \quad \text{as } t_2 \rightarrow t_1,$$

uniformly with respect to $(u, v) \in \overline{B}_R$. By the Arzela-Ascoli theorem, we deduce that $A(\overline{B}_R)$ is relatively compact, and then A is a compact operator. By the Schauder fixed point theorem, we conclude that operator A has at least one fixed point (u, v) in \overline{B}_R , which is a solution of our problem $(S) - (BC)$. \square

Theorem 6. Assume that (I1) and

(I6) The functions $f, g : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous and there exist $p_i \geq 0$, $i = 0, \dots, 4$ with at least one nonzero, $q_i \geq 0$, $i = 0, \dots, 4$ with at least one nonzero, and nondecreasing functions $h_i, k_i \in C([0, \infty), [0, \infty))$ $i = 1, \dots, 4$ such that

$$\begin{aligned} |f(t, x_1, x_2, x_3, x_4)| &\leq p_0 + \sum_{i=1}^4 p_i h_i(|x_i|), \\ |g(t, y_1, y_2, y_3, y_4)| &\leq q_0 + \sum_{i=1}^4 q_i k_i(|y_i|), \end{aligned}$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$,

hold. If there exists $\Xi_0 > 0$ such that

$$\begin{aligned} &\left(p_0 + p_1 h_1(\Xi_0) + p_2 h_2(\Xi_0) + p_3 h_3\left(\frac{\Xi_0}{\Gamma(\theta_1 + 1)}\right) + p_4 h_4\left(\frac{\Xi_0}{\Gamma(\sigma_1 + 1)}\right) \right) M_7 \\ &+ \left(q_0 + q_1 k_1(\Xi_0) + q_2 k_2(\Xi_0) + q_3 k_3\left(\frac{\Xi_0}{\Gamma(\theta_2 + 1)}\right) + q_4 k_4\left(\frac{\Xi_0}{\Gamma(\sigma_2 + 1)}\right) \right) M_9 < \Xi_0, \end{aligned} \quad (13)$$

then problem $(S) - (BC)$ has at least one solution on $[0, 1]$.

Proof. We consider the set $\overline{B}_{\Xi_0} = \{(u, v) \in Y, \|(u, v)\|_Y \leq \Xi_0\}$, where Ξ_0 is given in the theorem. We will prove that $A : \overline{B}_{\Xi_0} \rightarrow \overline{B}_{\Xi_0}$. For $(u, v) \in \overline{B}_{\Xi_0}$ and $t \in [0, 1]$ we obtain

$$\begin{aligned} |A_1(u, v)(t)| &\leq (p_0 + p_1 h_1(\Xi_0) + p_2 h_2(\Xi_0) \\ &\quad + p_3 h_3 \left(\frac{\Xi_0}{\Gamma(\theta_1 + 1)} \right) + p_4 h_4 \left(\frac{\Xi_0}{\Gamma(\sigma_1 + 1)} \right)) M_7, \\ |A_2(u, v)(t)| &\leq (q_0 + q_1 k_1(\Xi_0) + q_2 k_2(\Xi_0) \\ &\quad + q_3 k_3 \left(\frac{\Xi_0}{\Gamma(\theta_2 + 1)} \right) + q_4 k_4 \left(\frac{\Xi_0}{\Gamma(\sigma_2 + 1)} \right)) M_9, \end{aligned}$$

and then for all $(u, v) \in \overline{B}_{\Xi_0}$ we have

$$\begin{aligned} \|A(u, v)\|_Y &\leq \left(p_0 + p_1 h_1(\Xi_0) + p_2 h_2(\Xi_0) + p_3 h_3 \left(\frac{\Xi_0}{\Gamma(\theta_1 + 1)} \right) + p_4 h_4 \left(\frac{\Xi_0}{\Gamma(\sigma_1 + 1)} \right) \right) M_7 \\ &\quad + \left(q_0 + q_1 k_1(\Xi_0) + q_2 k_2(\Xi_0) + q_3 k_3 \left(\frac{\Xi_0}{\Gamma(\theta_2 + 1)} \right) + q_4 k_4 \left(\frac{\Xi_0}{\Gamma(\sigma_2 + 1)} \right) \right) M_9 \\ &< \Xi_0. \end{aligned}$$

Then $A(\overline{B}_{\Xi_0}) \subset \overline{B}_{\Xi_0}$. In a similar manner used in the proof of Theorem 5, we can prove that operator A is completely continuous.

We suppose now that there exists $(u, v) \in \partial B_{\Xi_0}$ such that $(u, v) = \nu A(u, v)$ for some $\nu \in (0, 1)$. We obtain as above that $\|(u, v)\|_Y \leq \|A(u, v)\|_Y < \Xi_0$, which is a contradiction because $(u, v) \in \partial B_{\Xi_0}$. Then by the nonlinear alternative of Leray-Schauder type, we conclude that operator A has a fixed point $(u, v) \in \overline{B}_{\Xi_0}$, and so problem $(S) - (BC)$ has at least one solution. \square

4. Examples

Let $\alpha = \frac{5}{2}$ ($n = 3$), $\beta = \frac{10}{3}$ ($m = 4$), $\theta_1 = \frac{1}{3}$, $\sigma_1 = \frac{9}{4}$, $\theta_2 = \frac{16}{5}$, $\sigma_2 = \frac{25}{6}$, $\gamma_0 = \frac{4}{3}$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{3}{4}$, $\delta_0 = \frac{11}{5}$, $\delta_1 = \frac{1}{6}$, $\delta_2 = \frac{15}{7}$, $H_1(t) = t^3$, $t \in [0, 1]$, $H_2(t) = \{0, t \in [0, \frac{1}{3}]; 2, t \in [\frac{1}{3}, 1]\}$, $K_1(t) = \{0, t \in [0, \frac{1}{2}]; 4, t \in [\frac{1}{2}, 1]\}$, $K_2(t) = t^2$, $t \in [0, 1]$.

We consider the system of fractional differential equations

$$(S_0) \quad \begin{cases} D_{0+}^{5/2} u(t) + f(t, u(t), v(t), I_{0+}^{1/3} u(t), I_{0+}^{9/4} v(t)) = 0, & t \in (0, 1), \\ D_{0+}^{10/3} v(t) + g(t, u(t), v(t), I_{0+}^{16/5} u(t), I_{0+}^{25/6} v(t)) = 0, & t \in (0, 1), \end{cases}$$

with the boundary conditions

$$(BC_0) \quad \begin{cases} u(0) = u'(0) = 0, & D_{0+}^{4/3} u(1) = 3 \int_0^1 t^2 D_{0+}^{1/2} u(t) dt + 2 D_{0+}^{3/4} u\left(\frac{1}{3}\right), \\ v(0) = v'(0) = v''(0) = 0, & D_{0+}^{11/5} v(1) = 4 D_{0+}^{1/6} v\left(\frac{1}{2}\right) + 2 \int_0^1 t D_{0+}^{15/7} v(t) dt. \end{cases}$$

We obtain $\Delta_1 \approx -0.83314732 \neq 0$ and $\Delta_2 \approx -0.85088584 \neq 0$. So assumption $(I1)$ is satisfied. In addition, we have $M_1 \approx 2.11984652$, $M_2 \approx 1.39227116$, $M_3 \approx$

1.12892098, $M_4 \approx 1.03231866$, $M_5 = M_1$, $M_6 = M_3$, $M_7 \approx 1.98819306$, $M_8 \approx 1.68729195$, $M_9 \approx 1.95523852$, $M_{10} \approx 1.84725332$.

Example 1. We consider the functions

$$\begin{aligned} f(t, x_1, x_2, x_3, x_4) &= \frac{1}{\sqrt{4+t^2}} + \frac{|x_1|}{7(t+1)^3(1+|x_1|)} - \frac{t}{8} \arctan x_2 \\ &\quad + \frac{t^2}{t+9} \cos x_3 - \frac{1}{2(t+10)} \sin^2 x_4, \\ g(t, y_1, y_2, y_3, y_4) &= \frac{2t}{t^2+9} - \frac{1}{10} \sin y_1 + \frac{|y_2|}{4(2+|y_2|)} + \frac{1}{12} \arctan y_3 - \frac{t}{t+20} \cos^2 y_4, \end{aligned}$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$. We find the inequalities

$$\begin{aligned} |f(t, x_1, x_2, x_3, x_4) - f(t, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)| &\leq \frac{1}{7} \sum_{i=1}^4 |x_i - \tilde{x}_i|, \\ |g(t, y_1, y_2, y_3, y_4) - g(t, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)| &\leq \frac{1}{8} \sum_{i=1}^4 |y_i - \tilde{y}_i|, \end{aligned}$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$. So we have $L_1 = \frac{1}{7}$, $L_2 = \frac{1}{8}$, and $\Xi \approx 0.878 < 1$. Therefore, assumption (I2) is satisfied, and by Theorem 1 we deduce that problem $(S_0) - (BC_0)$ has at least one solution $(u(t), v(t))$, $t \in [0, 1]$.

Example 2. We consider the functions

$$\begin{aligned} f(t, x_1, x_2, x_3, x_4) &= \frac{t+1}{t^2+3} (3 \sin t + \frac{1}{4} \sin x_1) - \frac{1}{(t+3)^2} x_2 + \frac{t}{4} \arctan x_3 - \cos x_4, \\ g(t, y_1, y_2, y_3, y_4) &= \frac{e^{-t}}{1+t^2} - \frac{1}{3} \sin y_2 + \cos^2 y_3 + \frac{1}{5} \arctan y_4, \end{aligned}$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$. Because we have

$$\begin{aligned} |f(t, x_1, x_2, x_3, x_4)| &\leq \frac{5}{2} + \frac{1}{8} |x_1| + \frac{1}{9} |x_2| + \frac{1}{4} |x_3|, \\ |g(t, y_1, y_2, y_3, y_4)| &\leq 2 + \frac{1}{3} |y_2| + \frac{1}{5} |y_4|, \end{aligned}$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$, then assumption (I3) is satisfied with $c_0 = \frac{5}{2}$, $c_1 = \frac{1}{8}$, $c_2 = \frac{1}{9}$, $c_3 = \frac{1}{4}$, $c_4 = 0$, $d_0 = 2$, $d_1 = 0$, $d_2 = \frac{1}{3}$, $d_3 = 0$, $d_4 = \frac{1}{5}$. Besides, we obtain $M_{11} \approx 0.805142$, $M_{12} \approx 0.885295$ and $\Xi_1 = M_{12} < 1$. Then by Theorem 2 we conclude that problem $(S_0) - (BC_0)$ has at least one solution $(u(t), v(t))$, $t \in [0, 1]$.

Example 3. We consider the functions

$$\begin{aligned} f(t, x_1, x_2, x_3, x_4) &= -\frac{1}{4} x_1^{3/5} + \frac{1}{2(1+t)} \arctan x_4^{2/3}, \\ g(t, y_1, y_2, y_3, y_4) &= \frac{e^{-t}}{1+t^4} - \frac{1}{3} |y_2|^{1/2} + \sin |y_3|^{3/4}, \end{aligned}$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$. Because we obtain

$$|f(t, x_1, x_2, x_3, x_4)| \leq \frac{1}{4}|x_1|^{3/5} + \frac{1}{2}|x_4|^{2/3}, \quad |g(t, y_1, y_2, y_3, y_4)| \leq 1 + \frac{1}{3}|y_2|^{1/2} + |y_3|^{3/4},$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$, then assumption (I5) is satisfied with $a_0 = 0$, $a_1 = \frac{1}{4}$, $a_2 = 0$, $a_3 = 0$, $a_4 = \frac{1}{2}$, $b_0 = 1$, $b_1 = 0$, $b_2 = \frac{1}{3}$, $b_3 = 1$, $b_4 = 0$, $l_1 = \frac{3}{5}$, $l_4 = \frac{2}{3}$, $m_2 = \frac{1}{2}$, $m_3 = \frac{3}{4}$. Therefore, by Theorem 5 we deduce that problem $(S_0) - (BC_0)$ has at least one solution $(u(t), v(t))$, $t \in [0, 1]$.

Example 4. We consider the functions

$$f(t, x_1, x_2, x_3, x_4) = \frac{(1-t)^3}{10} + \frac{e^{-t}x_2^3}{20(1+x_1^2)} - \frac{t^2x_3^{1/3}}{5},$$

$$g(t, y_1, y_2, y_3, y_4) = \frac{t^2}{20} + \frac{1-t^2}{25}y_1^2 - \frac{1}{30}y_4^{1/5},$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$. Because we have

$$|f(t, x_1, x_2, x_3, x_4)| \leq \frac{1}{10} + \frac{1}{20}|x_2|^3 + \frac{1}{5}|x_3|^{1/3},$$

$$|g(t, y_1, y_2, y_3, y_4)| \leq \frac{1}{20} + \frac{1}{25}|y_1|^2 + \frac{1}{30}|y_4|^{1/5},$$

for all $t \in [0, 1]$, $x_i, y_i \in \mathbb{R}$, $i = 1, \dots, 4$, then assumption (I6) is satisfied with $p_0 = \frac{1}{10}$, $p_1 = 0$, $p_2 = \frac{1}{20}$, $p_3 = \frac{1}{5}$, $p_4 = 0$, $h_1(x) = 0$, $h_2(x) = x^3$, $h_3(x) = x^{1/3}$, $h_4(x) = 0$, $q_0 = \frac{1}{20}$, $q_1 = \frac{1}{25}$, $q_2 = 0$, $q_3 = 0$, $q_4 = \frac{1}{30}$, $k_1(x) = x^2$, $k_2(x) = 0$, $k_3(x) = 0$, $k_4(x) = x^{1/5}$. For $\Xi_0 = 2$, condition (13) is satisfied because

$$\begin{aligned} & \left(p_0 + p_1 h_1(2) + p_2 h_2(2) + p_3 h_3 \left(\frac{2}{\Gamma(\theta_1 + 1)} \right) + p_4 h_4 \left(\frac{2}{\Gamma(\sigma_1 + 1)} \right) \right) M_7 \\ & + \left(q_0 + q_1 k_1(2) + q_2 k_2(2) + q_3 k_3 \left(\frac{2}{\Gamma(\theta_2 + 1)} \right) + q_4 k_4 \left(\frac{2}{\Gamma(\sigma_2 + 1)} \right) \right) M_9 \\ & \approx 1.96264 < 2. \end{aligned}$$

Therefore, by Theorem 6 we conclude that problem $(S_0) - (BC_0)$ has at least one solution $(u(t), v(t))$, $t \in [0, 1]$.

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